# II. YIELD AND FAILURE CRITERIA FOR ISOTROPIC MATERIALS

The restriction to isotropic conditions has strong consequences, which in general are quite helpful. For example, for isotropy the linear elastic range properties are completely specified by only two independent moduli or compliances. It will be found that the failure criteria also are completely specified by two independent strength properties. Anisotropic materials will turn out to require many more failure type properties than for isotropy.

The objective in this section is to obtain general failure criteria that apply to all homogeneous and isotropic materials, thus crossing over and including the different materials types. This would not be possible if nano or micro scale mechanisms were brought in since these almost always require restriction to a particular materials type. There is no inherent incompatibility in characterizing failure at the macroscopic scale (as done here) and in characterizing the initiation of local failure at the nano-scale or at any other sub-scale for a particular material. Infact, such combined information is mutually reinforcing.

The relevant two strength properties will be specified by the failure values in uniaxial tension and uniaxial compression, designated by T and C respectively. The T/C ratio scans across materials types. The case of very ductile metals has T equal to or only slightly less than C. Tough, ductile polymers have T/C ratios near values of 3/4 or 2/3 or thereabouts. Brittle polymers have T/C of about 1/2 or less, as are brittle metals. Ceramics have T/C of about 1/5 or slightly larger, or much less. Glasses can have T/C of 1/10 or less. No single materials type has an exclusive range of T/C values. It is interesting that the spectrum of T/C values tracks the trend from ductile to brittle materials. The smaller the T/C value, the more likely the material is to be brittle, rather than ductile. Much will be made of this physical characteristic later.

The Mises and Tresca failure (yield) criteria with T = C apply to very ductile metals, but function poorly for all other materials types. The Coulomb-Mohr form is the only general failure criterion intended for all materials types that has survived historically, Coulomb [1], Mohr [2]. It has proven to be completely inadequate, but it still merits its place in the history of scientific efforts. This criterion is given by

$$\frac{\sigma_1}{T} - \frac{\sigma_3}{C} \leq 1 \tag{1}$$

where $\sigma_1$ is the largest principal stress and $\sigma_3$ is the smallest. In principal stress space the form (1) is a six sided pyramid enclosing the origin. The extremely simple, linear form of (1) immediately reveals its inadequacy. Even the Mises criterion involves a quadratic form, and quadratic forms are needed to provide locally smooth, outwardly convex failure surfaces which are understood to be generally required for isotropic materials. Mohr was well aware of this deficiency, but nobody subsequently was able to modify the Coulomb-Mohr form in a successful manner and retain a basic, two property format. Surprisingly few other general, isotropic failure theories have been developed over the historical time span, and none with anything even approaching the recognition level of the Coulomb-Mohr theory. The website http://www.efunda.com gives a complete summary of the Coulomb-Mohr criterion, as well as the Mises, the Tresca (maximum shear stress), and the maximum normal stress criteria.

The recent failure program began development in 1997. Over the next ten years seven papers were published on the methodology. Each paper confronted the development of a different and important facet of the overall problem. The final paper in 2007 integrated all the preceding developments to form a complete theory of yielding and failure. This final paper also gave a general introduction to the topic and a list of historically relevant references. These seven papers are available from the journals shown below, and the most recent five can be downloaded in LLNL manuscript form from the FailureCriteria.com homepage.

Christensen, R. M., 2007, "A Comprehensive Theory of Yielding and Failure for Isotropic Materials", J. Engr. Mater. and Technol., 129, 173-181.

Christensen, R. M., 2006, "A Comparative Evaluation of Three Isotropic, Two Property Failure Theories", J. Appl. Mech., 73, 852-859.

Christensen, R. M., 2006, "Yield Functions and Plastic Potentials for BCC Metals and Possibly Other Materials", J. Mech. of Mats. and Structs., 1, 195-212.

Christensen, R. M., 2005, “Exploration of Ductile, Brittle Failure Characteristics Through a Two Parameter Yield/Failure Criterion”, Mater. Sci. and Eng., A394, 417-424.

Christensen, R. M., 2004, “A Two Property Yield, Failure (Fracture) Criterion for Homogeneous, Isotropic Materials”, J. Engr. Mater. and Technol., 126, 45-52.

Christensen, R. M., 2000, “Yield Fuctions, Damage States and Intrinsic Strength:, Math. Mech. Solids, 5, 285-300.

Christensen, R. M., 1997, “Yield Functions/Failure Criteria for Isotropic Materials:, Proc. R. Soc. London, A453, 11473-1491.

A brief outline of the method of development is as follows. Take $I_1$ as the first invariant of the stress tensor, namely its trace. Take $J_2$ as the second invariant of the deviatoric stress tensor. Perform a polynomial expansion up to terms of second degree, and express that as the possible failure criterion. The result is

$$\alpha I_1 + \beta (I_1)^2 + \gamma J_2 \leq 1 \tag{2}$$

where $\alpha, \beta, and\ \gamma$ are parameters. All historical efforts to derive general failure criteria used the condition that the isotropic material would not fail under compressive hydrostatic stress. That condition will be used here, and it requires that the $(I_1)^2$ term in (2) vanish, $\beta = 0$, giving

$$\alpha I_1 + \gamma J_2 \leq 1 \tag{3}$$

This then leaves just two failure parameters, $\alpha\ and\ \gamma$, to be determined. This is done by calibrating (3) such that it specifies failure when the uniaxial stress reaches the values T and C, the failure properties in uniaxial tension and compression.

In compact, nondimensional, tensor form, the criterion (3) becomes

$$\alpha\hat{\sigma}_{ii} + \frac{3}{2}(1+\alpha)\hat{s}_{ij}\hat{s}_{ij} \leq 1 \tag{4}$$

where nondimensional stress is

$$\hat{\sigma}_{ij} = \frac{\sigma_{ij}}{C} \tag{5}$$

and

$$\alpha = \frac{C}{T} - 1 \tag{6}$$

Symbol $s_{ij}$ is the deviatoric stress tensor. The uniaxial strength values T and C are taken such that

$$\frac{T}{C} \leq 1$$

$$\alpha \geq 0$$

The failure criterion (4) is quite successful for materials considered to be normally or nominally ductile, but it is less satisfactory for brittle materials. This suggests that another mode of failure may be operative, perhaps a competing fracture mode of failure. A different but related recognition of this problem may be seen conceptually as follows. Yield functions are generally considered to have corners. The form (4) has no such characteristic. By far the most likely physical explanation for corners would be that they result from the intersection of two different modes of failure, say a ductile one and a brittle one. In considering a second, brittle mode of failure, fracture is the obvious explanation. However, this possibility inevitably leads to a paradox. Fracture mechanics gives the response to the stress concentration at a crack or other physical disruption whereas for the homogeneous materials which are being considered here there are no such disruptions or discontinuities. The resolution of this seeming paradox is that below the scale of the homogeneity, there certainly are flaws and defects which could initiate fracture, and which would ultimately manifest themselves as failure on the macroscopic scale.

Propagating cracks in homogeneous and isotropic materials tend to assume the direction consistent with a Mode I response to the maximum tensile stress component, even though they may have received forced initiation as something else, Broberg [3]. Therefore, a Mode I type fracture criterion will be taken as the possible brittle range failure criterion. This then gives

$$\hat{\sigma}_1 \leq \frac{1}{1+\alpha} \qquad if \qquad \alpha \geq 1 \tag{7}$$

where $\sigma_1$ is the largest principal stress. The requirement $\alpha \geq 1$ for this fracture criterion to be operative can be deduced as follows. The failure criterion (4) is geometrically a paraboloid in principal stress space. The fracture criterion (7) is that of three planes normal to the three coordinate axes in principal stress space. It is at the value $\alpha = 1$ that the three fracture controlled planes are just tangent to the yield/failure paraboloid. For $\alpha > 1$ the three planes cut sections out of the paraboloid. For values of $\alpha$ very near to $\alpha = 1$ these sections are very small, but they increase in size as $\alpha$ increases. The much more elaborate details of this construction are given in the above noted papers. The meaning of the $(1+\alpha)$ term in (7) will be made apparent below where these relations are fully written out in dimensional form.

Finally, the two governing failure criteria (4) and (7) take the following forms when expressed in terms of components,

$$\left(\frac{1}{T}-\frac{1}{C}\right)(\sigma_{11}+\sigma_{22}+\sigma_{33})+$$
$$\frac{1}{TC}\left\{\frac{1}{2}\left[(\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2\right]+3\left(\sigma_{12}^2+\sigma_{23}^2+\sigma_{31}^2\right)\right\}\leq 1 \tag{8}$$

and

$$if \quad T\leq\frac{C}{2} \tag{9}$$

then also

$$\sigma_1\leq T \tag{10}$$

where $\sigma_1$ is the largest principal stress.

When T = C the criterion (8) becomes the Mises criterion, and the fracture criterion (10) is inoperative because of (9). The general failure criteria (4)-(7) or (8)-(10) are completely specified by the two failure properties, T and C. Awareness of the paraboloidal part, (8), of this failure criterion goes back at least as far as to Mises. The fracture part, (10), also has historical antecedents as the maximum stress form. The present general theory, which necessarily coordinates both of them through (9), is new. The failure modes distinguish fracture type brittle failure from a yielding type of strength followed by plastic flow (idealized here as elastic-perfectly plastic behavior). The failure criteria (8) – (10) are displayed in three dimensional form on the Failure Surface Graphics page. They are displayed in two dimensional form and compared with other criteria at http://www.efunda.com.

In the above 2004 paper (downloadable from the FailureCriteria.com homepage) an extensive comparison with experimental data is given, for brittle as well as ductile materials. The data cases are for the following materials types : metallic, polymeric, ceramic, and geological. The stress states of testing are of both two dimensional and three dimensional forms, with quite extensive data for all four materials cases. The comparisons between the data and the theoretical predictions, based upon the T and C values for each material, are realistic and satisfactory. There has always been and always will be a need for reliable failure data. This is not only for the purpose of validating (or disproving) theoretical predictions, but also as a permanent data base in its own right. In general, the combination of a robust theory and an acute data base can provide the kind of leverage needed for advancement.

The yield/failure criterion (4) or (8) takes the form of a paraboloid in principal stress space, Fig. 1.

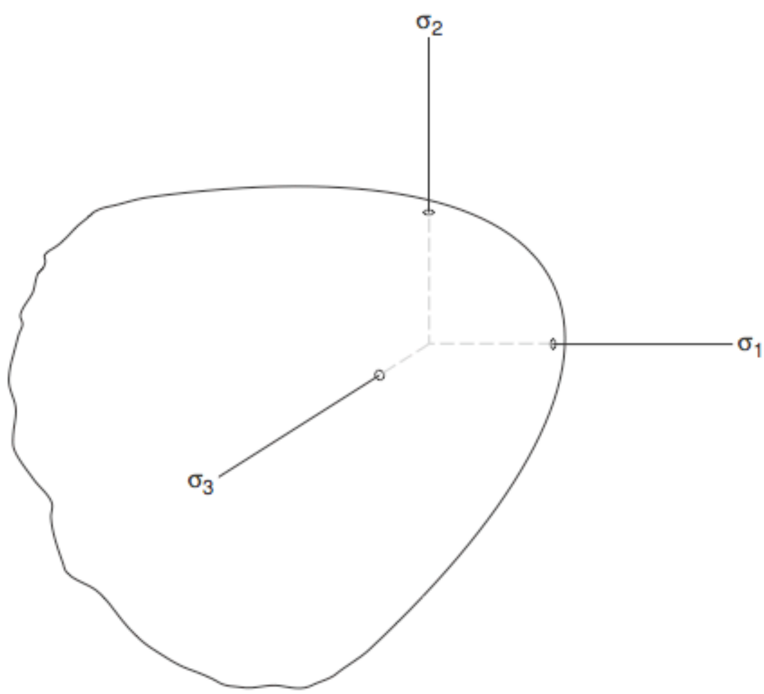


Fig. 1 Eq. (8), Failure Surface Paraboloid

As mentioned earlier, the fracture criteria (7) or (10) intersect the paraboloid and produce three flattened surfaces on it. This is most easily illustrated in two dimensional stress space $\sigma_{11}$ and $\sigma_{22}$ with the other stresses vanishing, Fig. 2, which is for T/C = 1/3.

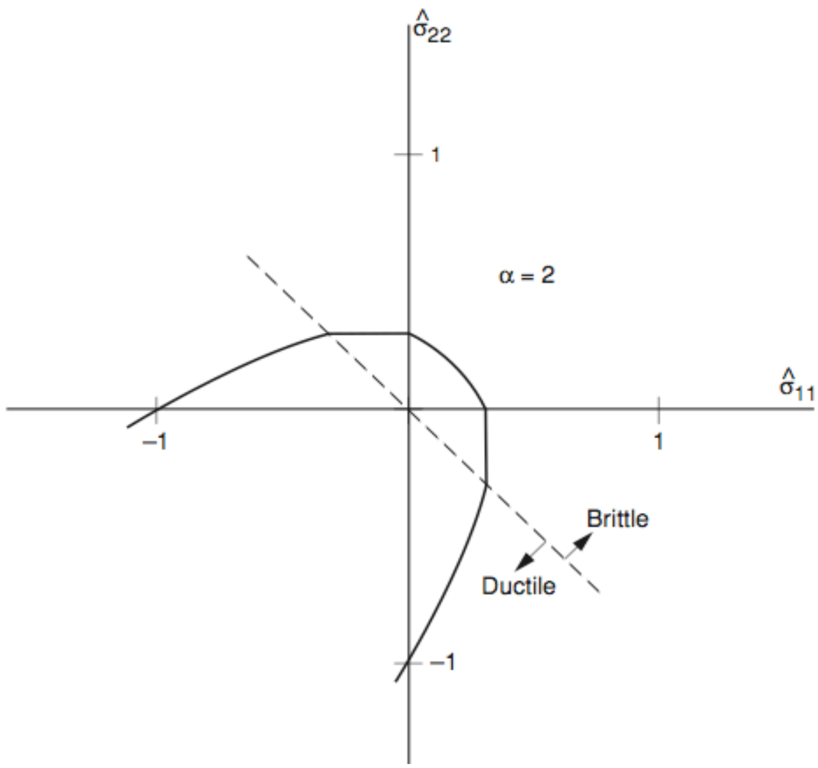


Fig. 2 Eqs. (8)-(10), Biaxial Stress, T/C=1/3

Referring to the biaxial stress state,the T/C = 1 case would be the usual Mises ellipse. At T/C = 1/2 the fracture criterion (10) is just tangent to the shifted and slightly smaller ellipse (8) at the intercepts with the axes. The T/C = 1/3 case, Fig. 2, is typical of cast iron and some ceramics, showing a very pronounced fracture cutoff effect. The limiting case T/C→0 reveals that applied stress can be sustained only if the mean normal stress is negative and no component of normal stress is positive.

The failure criteria (8)-(10) have been examined in considerable detail and developed into a criterion for ductile versus brittle behavior. Thus the modified paraboloid in stress space is subdivided into regions of brittle failure versus regions of ductile failure. The brittle regions are not just the fracture planes produced by (10) but also include portions of the paraboloid (8). For given values of T and C the specific stress state which is imposed determines whether the failure will be of ductile or of brittle nature. The resulting stress state division into ductile and brittle regions would not actually be expected to have a sharp dividing line between them, but rather to be of a transition zone nature in reality.

The ductile-brittle criterion is discussed and illustrated at length in the above papers. This ductile-brittle behavior can also be viewed for a particular stress state as showing the change as a function of the T/C variation. For example, for uniaxial tension it is found that

$$\frac{T}{C} > \frac{1}{2} \qquad \textit{Ductile}$$

$$\frac{T}{C} < \frac{1}{2} \qquad \textit{Brittle}$$

While for simple shear the result is

$$\frac{T}{C} > \frac{1}{3} \qquad \textit{Ductile}$$

$$\frac{T}{C} < \frac{1}{3} \qquad \textit{Brittle}$$

All stress states have T/C designated regions of ductile or brittle behavior. Some particular stress states are entirely of the ductile type for all values of T/C while some are entirely brittle.

The limiting case of the yield/failure paraboloid at $\alpha = 0$ is given by (4) or (8) as the Mises cylinder, Fig. 3.

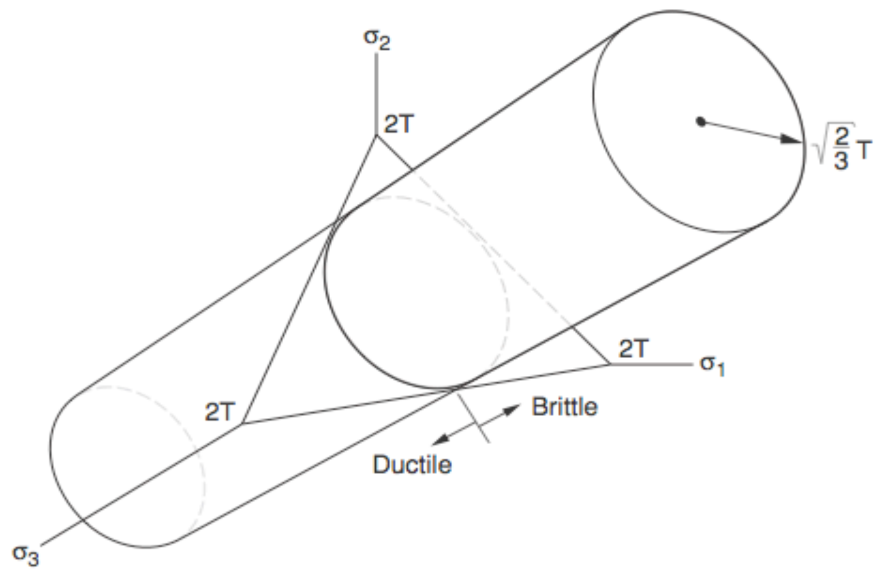


Fig. 3 Mises Cylinder, Principal Stress Space

The ductile-brittle criterion given in the papers divides the Mises cylinder into ductile versus brittle failure regions as shown. Most of the common stress states are on the ductile side of the division, but if the mean normal stress part of the stress state is sufficiently great, brittle failure will occur. This might seem to be a surprising result, but on further consideration it is to be expected. Consider first the effect of temperature. Temperature variation can determine whether a given material fails in a ductile or a brittle manner. So too can pressure control the ductile versus brittle failure character. Sufficient pressure can convert a nominally brittle material into a ductile material. Conversely, negative pressure can convert a nominally ductile material into a brittle material as shown in Fig. 3.

The development and consequences of this comprehensive yield/failure theory for isotropic materials have been examined in many different ways in the above papers. For applications where both yield strength and brittle failure are viewed together as generalized failure, the failure criteria (8)-(10) comprise the entire specification needed for analysis. The insertion of failure criteria (8)-(10) into finite element codes is especially easy to implement, use, and interpret.

Perhaps other new theories of yield and failure criteria for isotropic materials are yet to be developed. After they have received a reasonable level of recognition in the peer reviewed literature they will be surveyed here.

References

[1] Coulomb, C. A., 1773, "In Memories de Mathematique et de Physique," Academic Royal des Sciences par diver sans. 7 343-382.

[2] Mohr, O., 1900, "Welche Unstande Bedingen die Elastizitasgrenze und den Bruch eines Materials," Zeitschrift des Vereines Deutscher Ingenieure, 44, 1524-1530.

[3] Broberg, K. B., 1999, Cracks and Fracture, Academic Press, New York.

Richard M. Christensen
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