XXI. FAILURE THEORY/FAILURE CRITERIA FOR FIBER COMPOSITE LAMINATES

Richard M. Christensen and Kuldeep Lonkar

Introduction

It has been 50 years or more since the first serious efforts were initiated to develop failure criteria for fiber composite materials. There is no need to give a historical summary, the litany is well known and well understood. It is sufficient to say that the huge and enervating effort has cascaded across many technical generations but still there is no tangible, usable, reliable and verified failure criterion for fiber composite materials.

By far the most prominent recent effort on composites failure was the World Wide Failure Exercise (WWFE), Hinton and Kaddour [1]. Although it received considerable criticism, for example Christensen [2], it did serve a beneficial purpose. It helped to define the scope and boundaries of the problem and it certainly revealed the formidable difficulties associated with the failure problem.

The degree of difficulty is evidenced by an apparent embargo on general failure criteria papers by one of the composites journals. This is understandable in recognition of the dismal record of success for the field. On the other hand, it rather constitutes an admission of defeat. In effect it says that the problem is too difficult to solve or for reasons not at all understood it is impossible to be solved. Either way it is an enforced block to understanding the ultimate condition of load bearing behavior for composites, what we ordinarily simply call failure.

Is there really no expectation or even possibility for success in this quest? Is it a quixotic quest? Until very recently one would have been forced to say that it is extremely unlikely that there ever will be a reliable and realistic failure criterion for fiber composite materials. However, exactly the same thing could have been said about isotropic materials. After literally centuries of dedicated effort, there still was no general, physically based failure criterion for isotropy! That totally negative status for isotropy very recently was suddenly and drastically reversed. There now is a meaningful failure criterion for all isotropic materials and most importantly, it offers high promise for anisotropic, fiber composites.

The recent book by Christensen [3] lays out the comprehensive and validated theory for the failure of isotropic materials. After publication of the book six supporting papers were written and published. See Ref. [4] for these six references, for the history of the field, and for a general perspective on the subject. The present paper develops the corresponding and companion failure theory/failure criteria for fiber composites. This work is mainly aimed toward treating the failure of carbon fiber/polymeric matrix composites in the standard form of laminates.

When dealing with the failure of fiber reinforced materials one usually starts with the case of aligned fiber composites. That case was most fundamentally treated by Christensen [5]. When considering the failure of general laminates there are two limiting cases that contain and limit all other laminate configurations. These are the unidirectional form and the quasi-isotropic form. As already mentioned, the unidirectional case was completely treated in [5]. The other limiting case, the quasi-isotropic case, will be thoroughly treated here for failure.

In addition to being one of the two limiting cases of all possible laminate configurations, the quasi-isotropic form is the single most important laminate configuration because it has no weak directions of fiber orientation. It is the "pure" composite material form. It provides the most meaningful measures with which to compare and assess the properties of carbon fiber composites versus those of standard isotropic materials such as aluminum, magnesium, steel, titanium, etc.

The configuration and consequent properties of quasi-isotropy are usually rationalized through stiffness considerations rather than through strength. Some investigators have argued that the strength for a quasi-isotropic stiffness arrangement is actually anisotropic in strength in the plane of the laminate. This has been the source of considerable controversy. It is here taken that the strength property of the laminate is also quasi-isotropic when the stiffness character is rationalized to be quasi-isotropic. This reasoning relates and appeals to the strength as being a laminate level property rather than as solely a lamina level property.

Even though the strength for the laminate may be taken to be quasi-isotropic there still is a strong motivation to somehow relate that back to the lamina level strength properties. This may seem to be contradictory coming just after arguing that the strength is a laminate property, but it will be shown to be possible when viewing the strength problem across the various involved length scales.

The overall problem of strength is immensely involved and complex. It will be unfolded and examined, and then synthesized here through the progression of many sequential steps, ultimately arriving at the desired end point, a rational method for treating the failure of fiber composite laminates. The entire development will follow the strict guidelines of theoretical mechanics analysis. The organization will be composed of the following sequence of focus areas:

Unidirectional Lamina Failure
First Ply Damage, Total Failure of Laminates
Quasi-Isotropic Conditions for Elasticity and Failure
Failure of Quasi-Isotropic Laminates
Experimental Evaluation
Failure of Orthotropic Laminates
Conclusions

The first significant step will be to recall and summarize the strength properties of the unidirectional lamina configuration, the ideal but usually not directly usable fiber composite materials form.

Unidirectional Lamina Failure

The failure criteria of interest are those for highly anisotropic, unidirectional fiber composites. The restriction to the condition of high anisotropy applies to both stiffness and strength. This condition is appropriate for carbon fiber polymeric matrix composites. The explicit polynomial invariants method for the highly anisotropic conditions with transversely isotropic symmetry requires decomposition of the failure criterion into two parts, as derived in Refs. [3] and [5]. These are the fiber controlled criterion and the matrix controlled criterion.

The derivation of the failure criteria in [3] and [5] is fully three-dimensional. The matrix controlled part of it required an extremely careful and intricate derivation. The main part of the derivation was given in the paper [5] and it was crucially supported by the associated micromechanics analysis given in the book [3].

Fiber Controlled Failure

$$-C_{11} \leq \sigma_{11} \leq T_{11} \tag{1}$$

where axis 1 is in the fiber direction and T_{11} and C_{11} are the fiber direction tensile and compressive strengths. The matrix controlled failure criterion is given by

Matrix Controlled Failure

$$\begin{aligned}\left(\frac{1}{T_{22}}-\frac{1}{C_{22}}\right)(\sigma_{22}+\sigma_{33})+\frac{1}{T_{22}C_{22}}(\sigma_{22}+\sigma_{33})^2+\frac{1}{S_{23}^2}(\sigma_{23}^2-\sigma_{22}\sigma_{33})\\+\frac{1}{S_{12}^2}(\sigma_{12}^2+\sigma_{31}^2)\leq 1\end{aligned} \tag{2}$$

and where the transverse shear strength is specified by

$$S_{23}^2=\left(\frac{1+\frac{T_{22}}{C_{22}}}{3+5\frac{T_{22}}{C_{22}}}\right)T_{22}C_{22} \tag{3}$$

In principal stress space σ_2, σ_3 of the 2-3 plane the failure criteria (2) and (3) combine to give the especially simple single form for the matrix controlled failure criterion as

$$\begin{aligned}\left(\frac{1}{T_{22}}-\frac{1}{C_{22}}\right)(\sigma_2+\sigma_3)+\\\frac{1}{T_{22}C_{22}}\left[\sigma_2^2+\sigma_3^2-\left(\frac{1+3\frac{T_{22}}{C_{22}}}{1+\frac{T_{22}}{C_{22}}}\right)\sigma_2\sigma_3\right]+\frac{(\sigma_{12}^2+\sigma_{31}^2)}{S_{12}^2}\leq 1\end{aligned} \tag{4}$$

In (2), (3), and (4) T_{22} and C_{22} are the transverse tensile and compressive strengths, S_{12} is the axial shear strength, and S_{23} is the transverse shear strength.

As proven in Ref. [5] the transverse shear strength S_{23} is not an independent strength property but is determined by T_{22} and C_{22}. This is in complete consistency with the derivation of the isotropic materials failure, Ref. [3].

Thus this aligned fiber composites theory passes the first test of consistency. When isotropic materials disallow independent shear strength, then the transverse shear strength for transversely isotropic materials very likely must also not be independent but be determined by the other failure properties. That behavior occurs with (2)-(4).

The range of the T_{22} and C_{22} strength properties in (2)-(4) is given by

$$0 \leq \frac{T_{22}}{C_{22}} \leq 1 \tag{5}$$

Using (5) in (3) then requires

$$\frac{1}{4} \leq \frac{S_{23}^2}{T_{22}C_{22}} \leq \frac{1}{3} \tag{6}$$

or

$$0.5 \leq \frac{S_{23}}{\sqrt{T_{22}C_{22}}} \leq 0.577 \tag{7}$$

The values of T_{22}/C_{22} are usually about 1/4 to 1/3. The corresponding values for S_{23} are

$$\begin{aligned} &At\,\frac{T_{22}}{C_{22}} = \frac{1}{4} \qquad && S_{23} = 0.542\sqrt{T_{22}C_{22}} \\ &At\,\frac{T_{22}}{C_{22}} = \frac{1}{3} \qquad && S_{23} = 0.535\sqrt{T_{22}C_{22}} \end{aligned} \tag{8}$$

It would be virtually impossible to routinely determine S_{23} experimentally to the accuracy required by (7) and (8). S_{23} is a prediction from the theory and extremely difficult to determine experimentally as an independent entity.

This is the first and only derivation of failure criteria for unidirectional lamina that involves an independent prediction of the transverse shear strength, S_{23}, in (3). Commonly reported values for S_{23} fall on either side of (3), both larger and smaller. Unfortunately commonly reported values for S_{23} are notoriously unreliable. It will be highly advantageous to evaluate this theory of failure by comparing the prediction (3) for S_{23} against some certified and extremely reliable test data. It is not clear that such data are presently available. This remains a major priority for the future. Nevertheless, it can be said with certainty that the theoretical result (3) is consistent with the broad range of widely scattered test data available for S_{23}.

The unidirectional fiber composites failure criteria in (1) and (2) or (4) are very easy to use. The fiber controlled criterion (1) is identical with the common, intuitive form. For some this may seem too simple to be realistic, but that would be incorrect reasoning. It is the rigorous result that comes directly from the polynomial invariants theory method in the case of highly anisotropic fiber composites. The two failure criteria are the most rigorous forms available. They follow from a rational derivation rather than merely being postulated, as are most failure criteria for unidirectional lamina.

The failure criteria (1) and (4) are also the simplest ones since they only require 5 strength properties to calibrate them. Most failure criteria require a large number of parameters to be specified. This fiber composites failure theory is calibrated by only 5 strength properties, the same as the number of independent elastic properties for transverse isotropy. This is in complete harmony with the corresponding behavior of isotropic materials where only two properties suffice for each function, stiffness or strength.

First Ply Damage, Total Failure of Laminates

Having succeeded in developing the failure criteria for the unidirectional lamina, it is the logical next step to examine all the lamina in a laminate to see which one is the first to fail under a prescribed loading of the laminate. This is the widely applied first ply failure approach. With the lamina failure criteria decomposed into fiber controlled modes of failure versus matrix controlled modes of failure, then these same two conditions exist for the laminate as independent entities.

The matrix controlled first ply failure is easily found and the independent fiber controlled first ply failure also is easily found. In the laminate context the matrix controlled first ply failure is not actually failure of the laminate, it is only failure of the single lamina. Accordingly and henceforth first ply matrix controlled failure will always be referred to as first ply damage. This much is straightforward and internally consistent.

The situation with fiber controlled first ply failure is vastly more complex than the matrix controlled case. Typically laminates are examined in the biaxial stress space of σ_{11} versus σ_{22}, or σ_x versus σ_y. In this case the fiber controlled first ply failure for a quasi-isotropic laminate has the diamond shaped failure envelope as shown in Ref. [3]. Two of the four vertices form extremely acute angles. If one tries to consider this first ply failure as the failure envelope for the laminate, it is completely and notoriously unsuccessful. This much has been known and understood for a long time. Thus fiber controlled first ply failure is not a useful concept in general and it will be abandoned here for any serious use with failure characterization.

The continuing term first ply damage has real meaning in the laminate context when the source is the matrix failure at the lamina level. The general concept of damage is widely used and applied in the laminate context. It has evolved that damage in one form or another is the organized but empirical treatment of states approaching failure of the laminate. It follows that damage is a useful concept and a real condition in composites but it is not helpful in understanding the actual, explicit failure of the laminate. Other means are required to arrive at an explicit criterion for the total failure of the laminate.

The pursuit of a general approach for the failure of laminates has been the "holy grail" for 50 years or more. There have been so many false starts and ridiculous claims that the pursuit is beginning to seem beyond extremely difficult, perhaps it is only vanishingly possible. But with the success in determining the failure criteria for the unidirectional lamina, there is renewed hope for success in the laminate case. The most important new understanding at this point is the realization that fiber controlled first ply failure is <u>not</u> a successful approach for treating laminate failure. That is, it is not successful for treating the total failure of the laminate. After that realization is fully assimilated, a new outlook of promise emerges. This is embodied in the following failure theorem that refers to laminates under plane stress conditions with no out of plane loadings or delaminations.

Laminate Failure Theorem

There exist failure modes in the laminate that do not exist at the lamina level. Conversely there exist failure modes at the lamina level that do not exist at the laminate level.

The proof of this theorem will be by physically based, tangible examples. It would not be possible to prove this theorem by purely mathematical means.

In summary of the conclusions arrived at thus far:

(i) First ply damage in the laminate is caused by and on the level of lamina induced matrix controlled failure behavior.

(ii) Failure of the laminate is initiated at and induced at the laminate level even though there may be precursor events at the lamina level.

Quasi-Isotropic Conditions for Elasticity and Failure

As outlined in the Introduction section, the quasi-isotropic configuration for the arrangement of all lamina is by far the most important of all laminate configurations. This is obtained by arranging the lamina orientations at equal angles with a minimum of three separate orientations. The common form is the 0, 90, $\pm$45 degree set of orientations. In this section all the information needed to treat the failure of quasi-isotropic laminates will be assembled. The formal treatment of failure will be given in the following section. All forms are for plane stress conditions, as is normal.

The first characterization needed for the quasi-isotropic laminate is the in-plane elastic modulus E and the Poisson's ratio υ. These are given by

$$E = [(Q_{11} + Q_{22}) + 2Q_{12}]\frac{[(Q_{11} + Q_{22}) - 2Q_{12} + 4Q_{66}]}{[3(Q_{11} + Q_{22}) + 2Q_{12} + 4Q_{66}]} \tag{9}$$

and

$$\nu = \frac{[(Q_{11} + Q_{22}) + 6Q_{12} - 4Q_{66}]}{[3(Q_{11} + Q_{22}) + 2Q_{12} + 4Q_{66}]} \tag{10}$$

where at the lamina level

$$\begin{aligned} Q_{11} &= \frac{E_{11}}{1 - \nu_{12}\nu_{21}} \\ Q_{22} &= \frac{E_{22}}{1 - \nu_{12}\nu_{21}} \\ Q_{12} &= \frac{\nu_{12}E_{22}}{1 - \nu_{12}\nu_{21}} \\ Q_{66} &= \mu_{12} \end{aligned} \tag{11}$$

and

$$\nu_{21} = \frac{E_{22}}{E_{11}}\nu_{12}$$

and where axis 1 is in the fiber direction. It is believed that this is the first time that these simplest forms for quasi-isotropic E and υ expressed in terms of lamina properties have been derived and displayed.

Next the basic form for the failure criterion for the quasi-isotropic laminate must be identified or established. As already explained the first ply fiber failure approach is completely discredited. In view of the similarity of the quasi-isotropic state for the laminate and that for fully isotropic materials, the failure form for the latter will be taken to give the trial or provisional form for the former. In particular, the polynomial invariants method for isotropy will be stated here in appropriate form for the quasi-isotropy case of the laminate. This procedure has been fully developed in Ref. [3] and from Eq. (12.5) of that source the provisional failure criterion for the quasi-isotropic laminate is given by

$$\left(\frac{1}{T} - \frac{1}{C}\right)(\sigma_{11} + \sigma_{22}) + \frac{1}{TC}(\sigma_{11} + \sigma_{22})^2 + \frac{1}{S^2}(\sigma_{12}^2 - \sigma_{11}\sigma_{22}) \leq 1 \qquad (12)$$

where T and C are the uniaxial tensile and compressive strengths and S is the in-plane shear strength. Index notation will also be used here with the laminate case as it was with the lamina case in the second section. The difference will always be clear from the context.

Finally for this assembly of required forms, the conditions relating stresses at the lamina level and at the laminate level will be given. This will be of use in the failure theory development of the next section. The ply orientations for the quasi-isotropic laminate are as shown in Fig. 1.

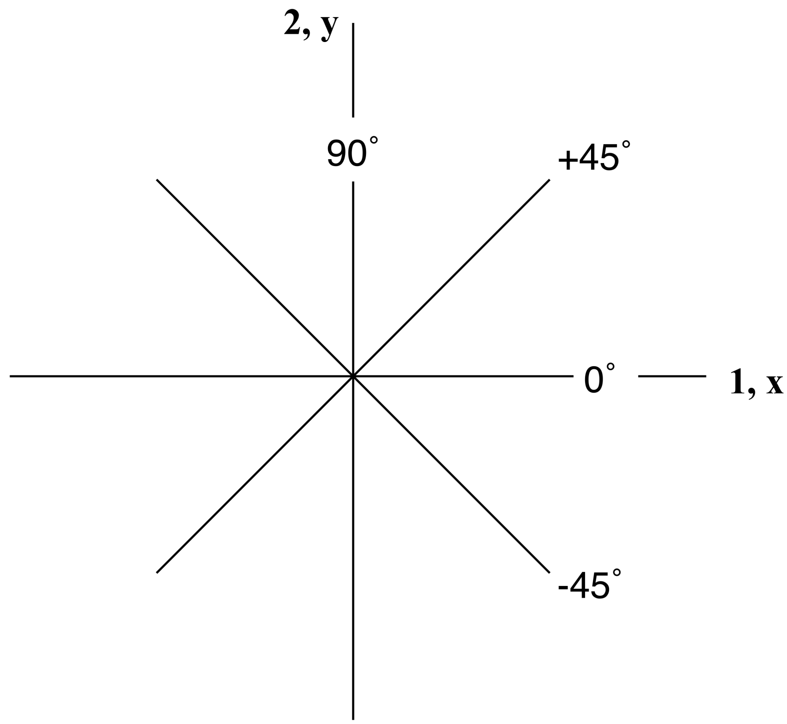

Fig. 1 Lamina orientations giving quasi-isotropy

0˚ Lamina

$$\sigma_{11} = \frac{E_{11}}{E(1 - \nu_{12}\nu_{21})}\left[(1 - \nu\nu_{21})\sigma_x + (\nu_{21} - \nu)\sigma_y\right] \qquad (13)$$

$$\sigma_{22} = \frac{E_{22}}{E(1-\nu_{12}\nu_{21})}\left[(\nu_{12}-\nu)\sigma_x + (1-\nu\nu_{12})\sigma_y\right] \tag{14}$$

$$\sigma_{12} = 0 \tag{15}$$

+45˚ Lamina

$$\sigma_{11} = \frac{(1-\nu)(1+\nu_{21})E_{11}}{2(1-\nu_{12}\nu_{21})E}\left(\sigma_x + \sigma_y\right) \tag{16}$$

$$\sigma_{22} = \frac{(1-\nu)(1+\nu_{12})E_{22}}{2(1-\nu_{12}\nu_{21})E}\left(\sigma_x + \sigma_y\right) \tag{17}$$

$$\sigma_{12} = \frac{(1+\nu)\mu_{12}}{E}\left(\sigma_x - \sigma_y\right) \tag{18}$$

Two similar sets of relations apply for the other two lamina orientations. To avoid confusion between lamina and laminate notations, σ_x and σ_y in (13) and (14) refer to the laminate level stresses for biaxial stress states.

Failure of Quasi-Isotropic Laminates

All of the discussions and results up to this point have been necessary preliminaries. Now is the time and place to develop the explicit failure criterion for the most important fiber composites laminate of all, the quasi-isotropic form.

Equation (12) of the previous section is the starting point. This failure condition is calibrated by the three failure stresses T, C, and S. A decisive step will now be taken and justified to express the shear strength S in terms of the uniaxial strengths T and C.

In the case of three dimensionally isotropic materials failure, as developed by the polynomial invariants method in [3], the shear strength is given by

$$S^2 = \frac{TC}{3} \tag{19}$$

This is the three dimensional result from three dimensional isotropy. But it also follows that it is valid in the two dimensional case of quasi-isotropy for laminates. This is because all three failure stresses □, C, and S are defined and determined in the two dimensional sub-space of three space. That the form (19) applies to both 3-D isotropy and the 2-D quasi-isotropy of laminates is a critical step forward.

Substituting (19) into (12) gives

$$\left(\frac{1}{T}-\frac{1}{C}\right)(\sigma_{11}+\sigma_{22})+\frac{1}{TC}(\sigma_{11}+\sigma_{22})^2+\frac{3}{TC}(\sigma_{12}^2-\sigma_{11}\sigma_{22})\leq 1 \qquad (20)$$

Expressing (20) in the usual biaxial state of σ_{11} and σ_{22}, or equivalently in terms of principal stresses results in

$$\left(\frac{1}{T}-\frac{1}{C}\right)(\sigma_{11}+\sigma_{22})+\frac{1}{TC}(\sigma_{11}^2-\sigma_{11}\sigma_{22}+\sigma_{22}^2)\leq 1 \qquad (21)$$

This incredibly simple form is the complete failure criterion for the quasi-isotropic laminate. Only the uniaxial strengths T and C are needed to calibrate it, these are the uniaxial strengths from laminate level testing.

The failure criterion (21) is the final result when the strengths T and C are directly measured from tests on the quasi-isotropic laminate. However, it would be extremely advantageous if T and C could be related to the strength results obtained directly from the uniaxial lamina specimens of the same material. Usually these are called the tow strengths when referring to fiber direction strengths. In the notation of the unidirectional lamina, these would be the fiber controlled strengths T_{11} and C_{11}. Now we embark on a further program to relate the quasi-isotropic laminate strengths T and C to the tow strengths T_{11} and C_{11}.

The first step is to relate laminate tensile strength T to the tow strength T_{11}. The lamina to laminate stresses are given by relations (13)-(18). In the lamina to laminate stress relation (13) let $\sigma_{11}=T_{11}$ and $\sigma_x=T$, $\sigma_y=0$ then giving

$$T = \frac{E(1 - \nu_{12}\nu_{21})}{E_{11}(1 - \nu\nu_{21})} T_{11} \tag{22}$$

With E and ν from (9) and (10) and ν_{12} from direct measurement on the lamina and ν_{21} from (11), then (22) supplies the desired relation predicting the uniaxial tensile strength of the quasi-isotropic laminate in terms of the lamina properties and the tow tensile strength. An implicit condition in this derivation of (22) is that the fiber controlled failure in one of the four fiber directions implies failure of the quasi-isotropic laminate. This will require experimental verification, to be considered in Section 6.

Now turn to the corresponding compressive strength problem. It will be found to be much more complicated than the tensile case. We could as a trial path follow exactly the same procedure in the compressive case as was just done in the tensile case but that would be found to not compare well with test data. This will be shown with a data example later. The reason for the difficulty is fairly clear. Compressive failure modes involve the formation of kink bands (fiber buckling) or fiber splitting or perhaps combinations thereof. These compressive failure modes in the laminate may be much different from those in the lamina testing because of the constraint imposed by the neighboring lamina on any one particular lamina and its failure. This problem does not occur with the tensile failure case.

There is only one compressive case where the constraint imposed by the neighboring lamina do not influence the failure. This is the case of eqi-biaxial compression. Then all the lamina are at incipient failure at the same loading and they do not provide any additional constraints against failure, they all fail simultaneously. Thus eqi-biaxial compressive failure will be used determine uniaxial failure C in terms of lamina C_{11}.

In the eqi-biaxial compression case as simulated by (13) take

$$\sigma_x = \sigma_y = \sigma \tag{23}$$

and take

$$\sigma_{11} = -\alpha C_{11} \tag{24}$$

where α is an unknown nondimensional scale factor that must be determined independently. The procedure for doing that will be developed a littler later in the derivation. Scale factor α will be carried along for now.

Substitute (23) and (24) into (13) giving

$$\sigma = -\frac{E}{E_{11}}\left[\frac{(1-\nu_{12}\nu_{21})}{(1-\nu)(1+\nu_{21})}\right]\alpha C_{11} \tag{25}$$

In the case of eqi-biaxial compression, the stress at failure in the laminate for $\sigma_{11} = \sigma_{22} = \sigma$ is given by (21) as

$$\sigma = T - C - \sqrt{T^2 - TC + C^2} \tag{26}$$

Eliminating σ between (25) and (26) and solving for C gives the result

$$C = \frac{E}{E_{11}}\left[\frac{(1-\nu_{12}\nu_{21})}{(1-\nu)(1+\nu_{21})}\right]\left[\frac{\frac{\alpha C_{11}}{T_{11}} + 2\frac{(1-\nu)(1+\nu_{21})}{(1-\nu\nu_{21})}}{\frac{2\alpha C_{11}}{T_{11}} + \frac{(1-\nu)(1+\nu_{21})}{(1-\nu\nu_{21})}}\right]\alpha C_{11} \tag{27}$$

Next these results will be specialized to the condition of extreme anisotropy at the lamina level which is appropriate to this complete treatment of carbon fiber/polymeric composites.

Limiting Case Extreme Lamina Anisotropy

With extreme anisotropy in both stiffness and strength for the unidirectional lamina, the Q_{11} term in (9) and (10) is much larger than all the other Q terms giving

$$E = \frac{Q_{11}}{3} \tag{28}$$

and

$$\nu = \frac{1}{3} \tag{29}$$

Then from (11) it follows that

$$\nu_{21} = 0 \tag{30}$$

and

$$Q_{11} = E_{11} \tag{31}$$

Combining (28) and (31) gives

$$E = \frac{E_{11}}{3} \tag{32}$$

Eqs. (29) and (32) are the quasi-isotropic elastic properties in terms of the tow elastic property. Relations (29) and (30) and (32) comprise the intermediate results needed to proceed.

Using (28)-(32) in (22) and (27) gives

$$T = \frac{T_{11}}{3} \tag{33}$$

and

$$C = \left(\frac{1 + \frac{3\alpha C_{11}}{4T_{11}}}{1 + \frac{3\alpha C_{11}}{T_{11}}} \right) \alpha C_{11} \tag{34}$$

Combining (33) and (34) then yields the ratio of the strengths as

$$\frac{C}{T} = 3 \left(\frac{1 + \frac{3\alpha C_{11}}{4T_{11}}}{1 + \frac{3\alpha C_{11}}{T_{11}}} \right) \alpha \frac{C_{11}}{T_{11}} \tag{35}$$

To proceed further the unknown scale factor α must be evaluated. Perfect behavior occurs when the lamina strengths give $C_{11}/T_{11} = 1$. This perfect behavior at the lamina scale must also require perfect behavior at the laminate scale, namely $C/T = 1$. Imposing these two conditions on (35) results in

$$3\alpha\left(\frac{1+\frac{3}{4}\alpha}{1+3\alpha}\right) = 1 \tag{36}$$

This has the solution for the scale factor as

$$\alpha = \frac{2}{3} \tag{37}$$

Lamina to Laminate Strengths

Using (37) in (34) gives the final result, along with (33), as

$$\begin{matrix}\text{Laminate}\\ \text{Failure}\\ \text{Properties}\end{matrix}\left\{\begin{aligned} T &= \frac{T_{11}}{3} \\ C &= \left(\frac{2+\frac{C_{11}}{T_{11}}}{1+\frac{2C_{11}}{T_{11}}}\right)\frac{C_{11}}{3}\end{aligned}\right\}\begin{matrix}\text{Tow}\\ \text{Failure}\\ \text{Input}\end{matrix} \tag{38}$$

Taking the ratio of these two failure properties has

$$\frac{C}{T} = \left(\frac{2+\frac{C_{11}}{T_{11}}}{1+\frac{2C_{11}}{T_{11}}}\right)\frac{C_{11}}{T_{11}} \tag{39}$$

It follows that

$$\frac{C}{T} \geq \frac{C_{11}}{T_{11}} \tag{40}$$

Thus in the laminate the disparity or imbalance between C and T is less than that in the lamina. The laminate has a beneficial smoothing effect over that in the lamina.

The two final relations exhibited in (38) are the major results allowing tow measured failure properties to predict the entire failure behavior for the quasi-isotropic laminate through (21). Many crucial but physically justified steps were involved in this completing deduction. Next these theoretical results will be compared with high quality experimental data. There are no adjustable parameters and only two measured strength properties are involved.

Experimental Evaluation

We consider a quasi-isotropic laminate composed of IM-7 or equivalent carbon fibers in a polymeric matrix. From Hexcell data sheets the tow strengths in tension and compression are about

$$\begin{aligned} T_{11} &= 2700 \text{ MPa} \\ C_{11} &= 1700 \text{ MPa} \end{aligned} \tag{41}$$

The ratio of these is

$$\frac{C_{11}}{T_{11}} = 0.630 \tag{42}$$

From the relations in (38), the laminate uniaxial tensile and compressive strengths are predicted from the tow properties to be

$$\begin{aligned} T &= 900 \text{ MPa} \\ C &= 660 \text{ MPa} \end{aligned} \tag{43}$$

with the ratio

$$\frac{C}{T} = 0.733 \tag{44}$$

Comparing the tow and composites laminate ratios in (42) and (44) it is seen that the laminate is considerably more balanced in its tensile and compressive strengths mismatch than is the tow level.

The full spectrum of biaxial failure stresses is given by (21) when the calibrating properties (43) are substituted into it. The shear strength is then predicted to be

$$S = 445 \text{ MPa}$$

and the eqi-biaxial tensile and compressive strengths are predicted as

$$\sigma = 1047 \text{ MPa } and - 567 \text{ MPa}$$

Before getting to the experimental data, it is helpful to compare the polynomial invariants theoretical predictions from (21) and (43) with the corresponding first ply fiber controlled failure that also comes from the tow strengths. The results are shown in Fig. 2.

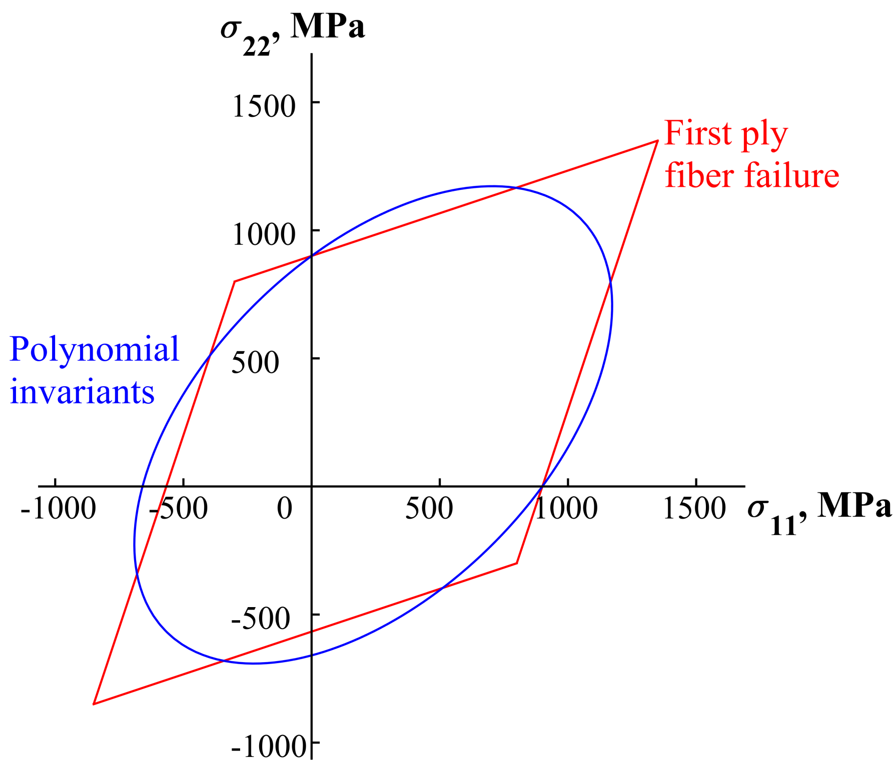

Fig. 2 Comparison between quasi-isotropic polynomial invariants failure theory, (21) and (43), and first ply fiber failure.

From Fig. 2 it is verified that the uniaxial tensile strengths by the two methods coincide but the uniaxial compressive strengths do not agree for the two methods. As explained fully in the derivation this is because of the constraints on compressive failure that occur in the laminate but do not occur in the lamina (tow) testing. The ridiculous predictions of the eqi-biaxial strengths (first and third quadrants) given by the first ply fiber failure criterion stand out in Fig. 2.

By one measure of position it can be shown that the polynomial invariants failure criterion ellipse in Fig. 2 is centered within the extent of the first ply fiber controlled failure envelope, the diamond shaped form in Fig. 2. This is an unexpected evidence of consistency.

Now the comparison between experimental failure data and the theory based predictions will be given. The results are shown in Fig 3. The data are taken as symmetrical about the 45˚ line in Fig. 3, Welsh et al [6]. Also the data in the first quadrant are not shown because they are highly doubtful for the following reason. The biaxial specimen type was that of a cruciform shape. Unfortunately this type of specimen has highly fracture prone stress concentrations at the interior corners under tension-tension conditions. The failure modes were said to be reasonable for conditions involving compressive stresses but no corresponding reassurance was offered for the tensile-tensile failure modes. The full data sets are shown in Refs. [3] and [6].

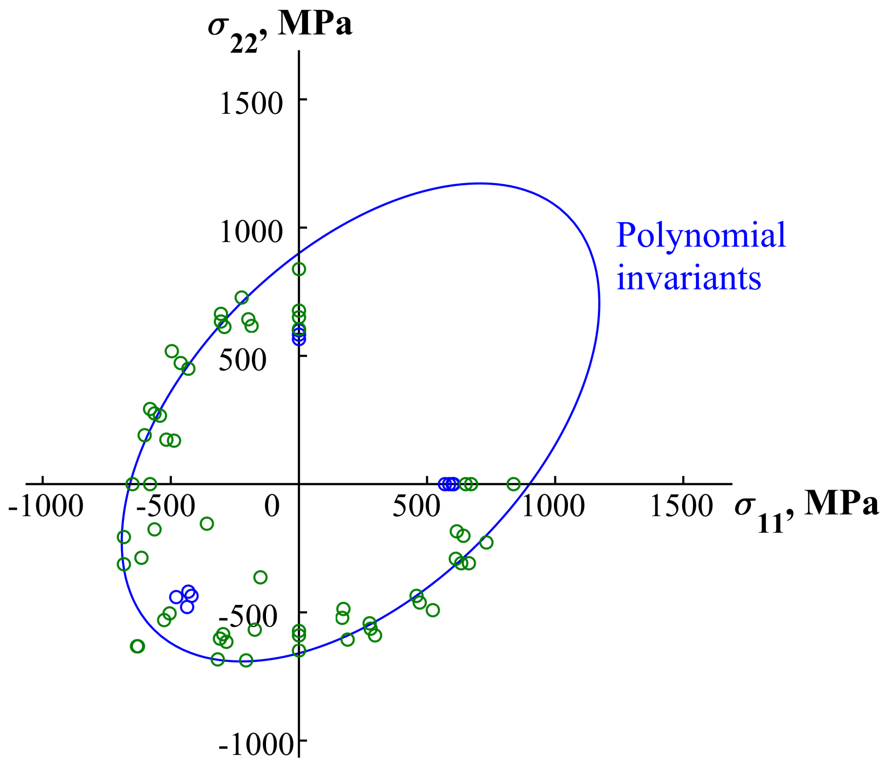

Fig. 3 Quasi-isotropic failure data, Welsh et al [6], versus the theoretical predictions from (21) and (43).

The theoretical versus data comparisons for eqi-biaxial compression and tension states are as follows:

	Theory	Data
Eqi-Biaxial Compression	-567 MPa	-550 MPa
Eqi-Biaxial Tension	1047 MPa	600 MPa

The data values for the two stress states are completely inconsistent. They suggest a Mises criterion with $C/T = 1$ which would be exceedingly unlikely for the quasi-isotropic laminate made of lamina with $C_{11}/T_{11} = 0.63$. It is the tensile data of the first quadrant data that are highly suspect. The uniaxial tensile data in Fig. 3 also understates the theoretical value for the same reason. The search for reliable tensile test data in the first quadrant will continue.

If the simple form $C = C_{11}/3$ were used instead of the correct form in (38) the compressive stress at eqi-biaxial failure would be $\sigma = -455$ MPa versus the theoretical prediction of -567 and the data value as -550.

It should also be mentioned that the theoretical predictions follow from the Hexcell data sheets for the tow properties of IM-7 type materials rather than the slightly different values given in [6].

Finally with respect to the theory vs. data comparison in Fig. 3, it is fairly good considering the typical scatter in the data. It is all the more relevant and useful considering that the theoretical prediction is completely calibrated by only the two tow level properties of the fiber composite material.

Dr. Welsh and his colleagues performed a commendable service for the discipline in developing the biaxial data generating system where the entire development required many years of testing and refinement. That is why there are so few reliable testing results under biaxial conditions for fiber composite materials. Further reliable failure data are sorely needed.

Failure of Orthotropic Laminates

With the success gained in the previous section on quasi-isotropic laminates, we extend the same polynomial invariants method to treat orthotropic laminates. From Christensen [3], Eq. (12.6) the polynomial invariants method for orthotropic symmetry yields the failure criterion as

$$\left(\frac{1}{T_{11}} - \frac{1}{C_{11}}\right)\sigma_{11} + \frac{\sigma_{11}^2}{T_{11}C_{11}} + \left(\frac{1}{T_{22}} - \frac{1}{C_{22}}\right)\sigma_{22} + \frac{\sigma_{22}^2}{T_{22}C_{22}} + \lambda_{12}\sigma_{11}\sigma_{22} + \frac{\sigma_{12}^2}{S_{12}^2} \le 1 \qquad (45)$$

where the T's and C's have the obvious identifications with uniaxial failure stresses and S_{12} is the shear strength. Parameter λ_{12} in (45) is a dimensional constant that requires further interpretation.

Write the 5th term in (45) as

$$\lambda_{12}\sigma_{11}\sigma_{22} = \beta\left(\frac{\sigma_{11}}{\sqrt{T_{11}C_{11}}}\right)\left(\frac{\sigma_{22}}{\sqrt{T_{22}C_{22}}}\right) \tag{46}$$

where the conventional notations of the 2nd and 4th terms in (45) have been employed. Now parameter β is of a nondimensional character. Determine β such that when the orthotropic form in (45) and (46) is reduced to quasi-isotropy it then must give the herein derived form in (20). This requires

$$\beta = -1 \tag{47}$$

Finally (46) and (47) incorporated back into (45) gives the final form

$$\begin{gathered}\left(\frac{1}{T_{11}}-\frac{1}{C_{11}}\right)\sigma_{11}+\left(\frac{1}{T_{22}}-\frac{1}{C_{22}}\right)\sigma_{22}+\left(\frac{\sigma_{11}}{\sqrt{T_{11}C_{11}}}-\frac{\sigma_{22}}{\sqrt{T_{22}C_{22}}}\right)^2 \\ +\left(\frac{\sigma_{11}}{\sqrt{T_{11}C_{11}}}\right)\left(\frac{\sigma_{22}}{\sqrt{T_{22}C_{22}}}\right)+\frac{\sigma_{12}^2}{S_{12}^2} \le 1\end{gathered} \tag{48}$$

For the general orthotropic layup of the fiber reinforced lamina, the failure form (48) is amazingly simple. The failure properties that calibrate the failure criterion (48) are

$$T_{11}, C_{11}, T_{22}, C_{22}, \& \ S_{12}$$

All of these are directly found from standard one-dimensional tests of strength determination. These five strengths for calibration are of the same number as the number of elastic constants required to calibrate the elastic behavior.

It would be unlikely that these five calibrating strength terms could be related to the tow properties in the way that was possible for the quasi-isotropic laminate, at least not with the same degree of rigor as was enforced there. This is not really a disadvantage. Determining the five strength properties by direct testing is completely approachable and reasonable.

A restriction must go along with the failure criterion (48). If one tries to reduce (48) to the case of a unidirectional form (lamina) it can be made to successfully recover the fiber controlled failure criterion (1) but it does not recover the matrix controlled failure criterion (2). To recover both, the approach must revert initially to the two decomposed forms (1) and (2). Relation (48) does not give both criteria.

The resulting and accompanying restriction appended to (48) must examine the three terms

$$\sqrt{T_{11}C_{11}},\ \ \sqrt{T_{22}C_{22}},\ \ \&\ \sqrt{3}S_{12} \tag{49}$$

None of the three terms in (49) can be an order of magnitude or more larger than either of the other two terms. Otherwise the failure criterion must be decomposed into fiber controlled versus matrix controlled modes of failure. This is not a serious restriction for the types of orthotropic layups normally and ordinarily employed.

An example will illustrate the use of the failure criterion (48). Take the calibrating failure properties as

$$\begin{aligned} T_{11} &= 1500 \text{ MPa} \\ C_{11} &= 1000 \text{ MPa} \\ T_{22} &= 500 \text{ MPa} \\ C_{22} &= 300 \text{ MPa} \\ S_{12} &= 200 \text{ MPa} \end{aligned} \tag{50}$$

These properties roughly correspond to taking 55% of the lamina in the σ_{11} direction and 15% each in the other 3 directions of the 4 direction layup pattern in Fig. 1.

Taking $\sigma_{12} = 0$ then (48) has the failure envelope as shown in Fig. 4. The predominate fiber reinforcement being in the σ_{11} direction has a strongly distorting effect on the size and shape of the failure envelope compared with that in Fig. 3 for quasi-isotropy.

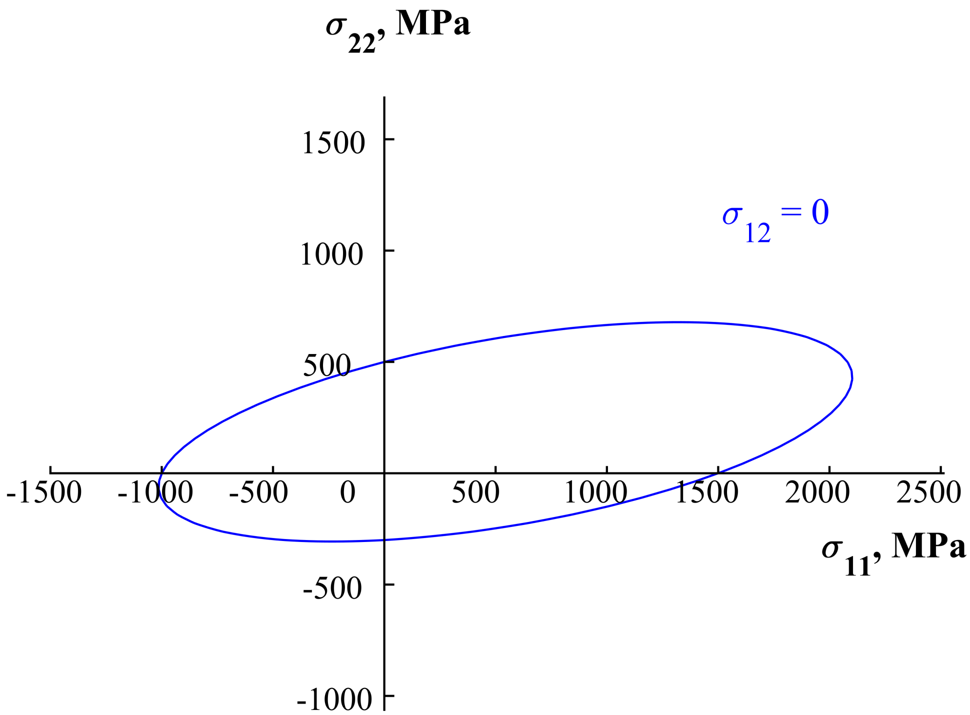

Fig. 4 Orthotropic failure envelope (48) for properties (50) with $\sigma_{12} = 0$.

The Fig. 4 failure envelope applies only when $\sigma_{12} = 0$. When $\sigma_{12} \neq 0$ then the failure envelopes for σ_{11} and σ_{22} are quite different. Also σ_{11} vs. σ_{22} vs σ_{12} can be visualized as a three dimensional failure envelope.

Since the quasi-isotropic form is a special case of orthotropic symmetry, its evaluation with experimental data in Section 6 also applies here for general orthotropy.

Conclusions

The two most important and also limiting cases of fiber composite laminates are the unidirectional form and the quasi-isotropic form. The failure criteria for the former are reviewed here and for the latter they are derived here in full detail.

It is a remarkable concurrence of results from this theory for the fiber dominated quasi-isotropic laminate that the elastic modulus E of the laminate is found to be one third that of the elastic modulus of the tow material (32), and the uniaxial tensile strength T of the laminate also is one third that of the tensile strength of the tow material (33). These exceptionally simple results are of basic and broad usefulness and enable a complete failure characterization.

The extremely close and tight relationship between quasi-isotropic laminates and three dimensional isotropic materials shows that all the testing validations for the 3-D isotropic materials case in [3] are strongly reinforcing to the validity of this quasi-isotropic fiber composites failure formalism, in addition to the data explicitly shown here. And that in turn reinforces the orthotropic failure formalism arrived at by the same polynomial invariants methodology.

Equally important as the quasi-isotropic results is the resulting failure criterion (48) for the general orthotropic case, of which quasi-isotropy is a special case. For the general orthotropic laminate case, the failure criterion is calibrated by five laminate failure properties, T_{11}, C_{11}, T_{22}, C_{22}, and S_{12}. These are determined directly from laminate testing.

The T_{11} and C_{11} failure properties for the orthotropic laminate must not be confused with the two tow (unidirectional) failure properties T_{11} and C_{11} for the quasi-isotropic case. The orthotropic case covers an enormous range of laminates for all the common and conventional layup patterns. Even just its existence is reassuring to the viability of treating failure in a rational manner.

Over the years the search for failure criteria for fiber composites has largely degenerated into curve fitting operations. Examples are from the World Wide Failure Exercise, Refs. [7]-[10], they illustrate that particular approach and its complications and consequences. All the present results could not have been derived without using the full arsenal of methods and techniques from the mechanics of materials behavior. Mechanics was the key, nothing was postulated and only failure properties measured directly from the fiber composite material were utilized.

Having herein derived and developed and validated the tensorially correct and complete forms for the failure criteria, the basic framework for the failure theory and the associated failure criteria is considered to be finished and completed. It now is a straightforward matter to build into this basic framework other ancillary aspects of failure such as residual stresses formed in processing, edge effects, delamination, and any of the other defects and complications commonly encountered.

It can now safely be said that there most certainly has been progress on failure theory over the most recent years. A new plateau has been reached for understanding composite materials failure behavior after 50+ years of travail. There now is a very promising future for the broad and reliable usage of failure criteria for carbon fiber/polymeric matrix composite materials.

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Richard M. Christensen
February 3rd, 2017