XVIII. A NEW THEORY OF STRAIN HARDENING AND ITS CONSEQUENCES FOR YIELD STRESS AND FAILURE STRESS

Introduction

It is almost as though the entire technical field gave up on trying to define yield stress and failure stress in the critical condition of uniaxial tension. There is no agreement on meaningful physical definitions of either, only an arbitrary offset rule that is often used with metals. As the result, all compilations of materials properties data have no guiding standards. Strength is often listed as ultimate strength, meaning the final fragmentation into pieces. Yield seems to have something to do with the end of the range of elastic behavior but not much more than that. Both are vague and loose concepts.

In recent work Christensen [1] devoted an entire chapter in a new book on materials failure to the subject of defining yield stress and failure stress. The present work takes that preceding examination much further.

The traditional approach separates failure into ductile versus brittle types without using a rigorous treatment of what ductility really means. The brittle case is taken to have yield stress coincident with immediate failure. So all attention focused upon the much more complex ductile case, more specifically the behavior of very ductile metals. The common approach was to separate the behavior into the elastic range and the plastic range. Yield stress was taken to provide a separation between the two ranges of behavior. The elastic range would switch over to the strain hardening plastic range, which would typically extend to large strains, sometimes even very large strains.

The ensuing complication was and is that at very large deformations in uniaxial tension the actual stress at failure is hardly indicative of a general measure of strength. Molecular, nanoscale changes in the material architecture at very large deformations make the results peculiar to uniaxial tension and by no means transferrable through a failure theory to behavior in other stress states. This leaves the definition of failure stress in an ambiguous state.

Failure stress (strength) would typically be taken as the limit of the plastic range. In trying to understand strength, the focus was always taken upon the strain hardening plastic range. It was taken as an obvious truth that failure could be understood and quantified only if the strain hardening range were to be understood in enough detail to lead to the explicit understanding of its termination by breakage, fragmentation.

That point of view prevailed decade after decade, century after century until finally a new, a drastically different reasoning was brought forward. Christensen [1] argued that failure most fundamentally represents the termination of the elastic range of behavior, not the following plastic range. The plastic range of behavior is "real" and very important, but it is not the key to understanding failure. The plastic range merely represents the transition from the elastic range to the failure state, no more, no less. This new point of view completely changes the theoretical approach to describing and defining failure, Ref. [1].

With this new point of view on failure, it is still found to be necessary to understand the strain hardening range of plastic behavior. This may seem to be contradictory after what has just been said but actually it is not. Strain hardening must be understood in its relationship to the preceding elastic behavior in order to properly define yield stress. And the rigorous definition of yield stress will be found to be necessary in order to properly define failure stress. So the following account will detail the critical examinations of the three separate and distinct topics of (i) strain hardening, (ii) yield stress, and (iii) failure stress. This account will begin by developing a new theory of strain hardening.

Strain hardening has always been treated empirically. Sometimes this range of the stress-strain relation has been taken in a power law form with the exponent to be determined and sometimes simply in parabolic form. Corresponding plasticity theories have been developed, Hill [2], Kachanov [3], and Lubliner [4]. The limiting case of strain hardening is that of elastic-perfectly plastic behavior, Prager and Hodge [5]. Strain hardening is certainly more complex than elastic behavior or elastic-perfectly plastic behavior and perhaps that is why it has only been treated empirically, with the apparent assumption that it is too difficult to be more specific. The present work will prove otherwise.

The single stress state of uniaxial tension has always been taken to be by far the most important single stress state. For all historical purposes that was the state for which data were and are reported and properties assigned. Uniaxial tensile properties are also of prime importance in constructing failure theory. It provides the "cornerstone" for calibrating the failure theory. Accordingly the present account will focus exclusively on the state of uniaxial tension for all matters of ductility, strain hardening, yield stress and failure stress. Computational applications under three-dimensional conditions and related open questions for all these matters will be discussed in the final section.

A New and Basic Theory of Strain Hardening

Elastic-plastic deformation of very ductile metals is usually and initially viewed as that of elastic-perfectly plastic deformation. This highly idealized state is as shown in Fig. 1.

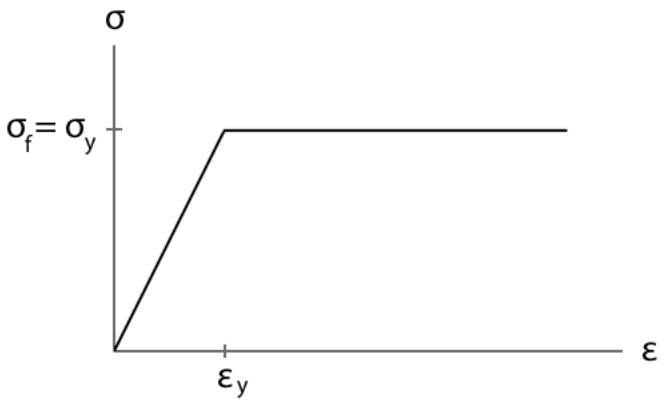

Fig. 1 Elastic-perfectly plastic solid

The yield stress and the failure stress are coincident and the yield strain is correspondingly specified, failure strain is not specified.

The discontinuity of modulus shown in Fig. 1 is never actually realized, such extreme behavior is not physically possible in engineering materials. Physical reality always rounds the corner shown in Fig. 1 to be more like the form shown in Fig. 2.

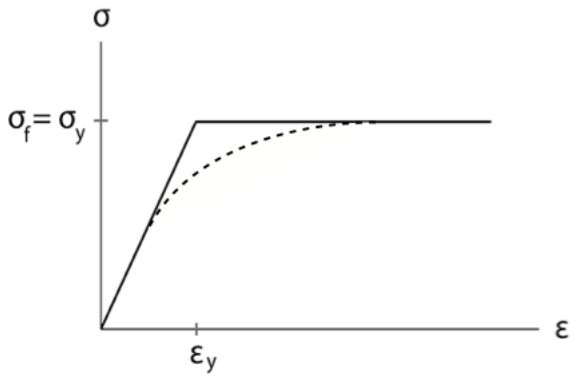

Fig. 2 More realistic behavior for σ vs. ε

In this case the yield stress and the failure stress become differentiated. More specifically, now take the still idealized stress-strain form as in Fig. 3.

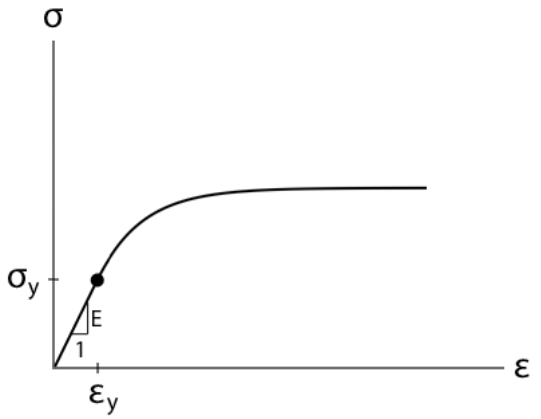

Fig. 3 Elastic, strain hardening behavior

The elastic modulus E is designated in Fig. 3 and the yield stress σ_y is taken as that at the terminus of the linear elastic range. The overall idealization is still taken as that appropriate to very ductile metals.

To proceed further a quit specific assumption will be invoked. For these homogeneous and isotropic materials, the stress-strain form is taken to

asymptotically approach a limiting failure stress σ_f consistent with the elastic range ultimately being terminated by failure, after the strain hardening.

The elastic range is specified by either of the forms

For $\varepsilon < \varepsilon_y$

$$\sigma = E\varepsilon$$

$$\frac{d\sigma}{d\varepsilon} = E \tag{1}$$

The second form will have an interesting and unexpected relationship to the strain hardening form to be found later. At this point the exact nature of the yield stress and yield strain has not been specified. That will carefully be defined later. For present purposes the perfectly elastic region is differentiated from the strain hardening region by (1).

Next the strain hardening region must be brought into consideration. The strain hardening region of elastic-plastic behavior has always been treated empirically. Usually it has been taken in a power law form between σ and ε but sometimes it has been given a parabolic form. The essential purpose here is to avoid making any particular assumption about its form but rather to derive it from basic considerations. While this possibility may seem to be unlikely considering the intense attention that has been expended upon this problem for so many years, there still remains a new approach for the problem, as will now be given.

The plastic deformation in highly ductile metals is caused by the flow of dislocations. As a dislocation moves through one atomic spacing in the crystal lattice, the atomic bond is ruptured with one adjacent atom and reformed with the next atom. All this transpires a stress level high enough to overcome the energy barrier holding the atoms in place.

There is another class of problems that involves a similar sequence of bond breakage followed by the formation of new bonds. This field is that of chemical reaction rate kinetics, Connors [6]. Reaction rate kinetics are found to be controlled by first order differential equations and they have

proven to be very successful. The counterpart of reaction rate kinetics is taken here for the strain hardening behavior in elastic-plastic solids.

The strain hardening region is specified in the form corresponding to 1st order reaction rate kinetics as

For $\varepsilon \geq \varepsilon_y$

$$\frac{d\sigma}{d\varepsilon} = \alpha\left(\sigma_f - \sigma\right) \tag{2}$$

The increment of increasing stress is due to an increment of increasing strain in proportion to $(\sigma_f - \sigma)$. Thus $(\sigma_f - \sigma)$ is the driving force for the strain hardening effect with σ_f being the failure stress. Parameter α in (2) is as yet unspecified. Parameter α is some nondimensional property of the material and it is yet to be determined, perhaps in terms of the other properties of the problem.

This 1st order ordinary differential equation (2) is taken as the balance law controlling the strain hardening as it progresses with increasing strains. The starting information for the problem is the elastic modulus, the yield stress, and the failure stress, as

$$E,\ \sigma_y,\ and\ \ \sigma_f$$

The given yield strain is then

$$\varepsilon_y = \frac{\sigma_y}{E}$$

The initial conditions are specified by

At $\varepsilon = \varepsilon_y$

$$\sigma = \sigma_y \tag{3}$$

$$\frac{d\sigma}{d\varepsilon} = E \tag{4}$$

The complete strain hardening solution of (2) satisfying (3) and (4) is given by

$$\sigma = \sigma_y + \sigma_0 \left[1 - e^{-\alpha\left(\varepsilon - \varepsilon_y\right)} \right] \tag{5}$$

where

$$\sigma_0 = \sigma_f - \sigma_y \tag{6}$$

$$\alpha = \frac{E}{\sigma_0} \tag{7}$$

Thus parameter α given by (7) controls the rate at which the strain hardening stress-strain curve approaches the asymptote σ_f.

There is a crucial observation to be made concerning the solution (5). The key step in the derivation is the knowledge that the finite failure stress exists. The failure stress σ_f doesn't have to be specified in value, only its existence is required. From the solution (5)-(7) σ_0 and thereby σ_f can be determined by a data point σ_1, ε_1 on the stress-strain record. From (5)-(7)

$$\frac{\sigma_0}{E} Ln\left[1 - \left(\frac{\sigma_1 - \sigma_y}{\sigma_0} \right) \right] = \varepsilon_y - \varepsilon_1 \tag{8}$$

After solving this for σ_0 and then σ_f from (6) the complete strain hardening form for a particular material is known. An example will be given later but the solution (5) has all the usual characteristics of standard strain hardening data forms.

This theory of strain hardening started with the idealized behavior of highly ductile metals. But the resulting solution (5) is not limited to just the dislocation sourced plasticity of very ductile metals. This is because failure itself is an intrinsic property of any isotropic material and it is intimately related to the elastic range of behavior. Failure is not controlled by the usual plasticity motivated facets of behavior. Plasticity itself does not dictate and control failure, it is but the interim path between elasticity and failure. It is the separate entities of elasticity and failure that are primal.

Combining the two solution forms (6) and (7) with (2) gives

$$\frac{d\sigma}{d\varepsilon} = E\left(\frac{\sigma_f - \sigma}{\sigma_f - \sigma_y}\right) \tag{9}$$

This controlling form (9) for strain hardening is even more transparent than is the starting form (2), especially when compared with the elasticity form (1b). Relation (9) embodies the complete behavior involved in going over the range $\sigma_y \le \sigma \le \sigma_f$. The solution (5) follows directly from (9). If stress is nondimensionalized by E then (9) takes a remarkably simple nondimensional form.

No matter what the material type may be, the solution of the first order differential equation (9) represents the breakage and in some cases the reformation of atomic bonds. It not only applies to highly ductile metals, it also applies as the strain hardening solution for all isotropic materials up to the point of strains large enough to begin introducing other extraneous nonlinear effects such as anisotropic effects due to grain alignments and molecular orientations in the loading direction.

Finally, one further observation is relevant. The controlling form (9) is the strain hardening counterpart of 1st order reaction rate kinetics. Second order reaction rate kinetics would involve raising the right hand side parenthesis term in (9) to the power 2, Connors [6]. Yet higher order forms follow similarly. Although such forms could be easily solved, it is believed that the first order form is that which applies to the strain hardening problem. The higher order forms would move toward the elastic-perfectly plastic case.

This strain hardening form for uniaxial tension can be converted to that for other states of stress using three dimensional plasticity theory.

Yield Stress as a 3rd Order Transition

Next a deeper look at yield stress will be undertaken. How should yield stress be formally defined? In the previous section that task was deferred until later. It was merely said that yield stress separates the purely elastic region from the strain hardening region.

The strain hardening solution (5)-(7) will now be used to be more specific about the meaning of the yield stress. Transitions have an important role to play in materials science. In the sense given by Ehrenfest the freezing of a liquid to become a solid is a 1st order transition, that of phase change. This involves the change of the free energy. An example of a 2nd order transition would be that of a martensitic transformation for steel. In the present context, the discontinuous change of the tangent modulus for the elastic-perfectly plastic solid of Fig. 1 would also be a 2nd order transition, if it were to occur, but which does not happen.

Going one step further from the constitutive form (1) in the elastic region and (5)-(7) for the strain hardening region, it is found that

For $\varepsilon < \varepsilon_y$

$$\frac{d^2\sigma}{d\varepsilon^2} = 0 \tag{10}$$

For $\varepsilon \geq \varepsilon_y$

$$\frac{d^2\sigma}{d\varepsilon^2} = -\frac{E^2}{\sigma_0} e^{-\frac{E}{\sigma_0}\left(\varepsilon - \varepsilon_y\right)} \tag{11}$$

Fig. 4 schematically shows the behavior of the second derivative in (10) and(11).

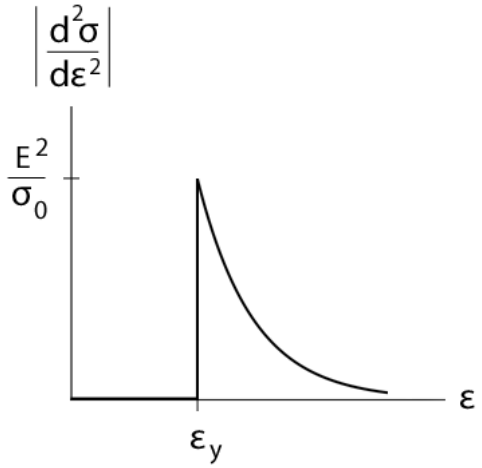

Fig. 4 Discontinuity in second derivative of stress-strain

It is seen from Fig. 4 that the material has undergone a classical 3rd order transition in traversing from the elastic state and region to the strain hardening state and region.

Thus in the present theory yield stress is a 3rd order transition. This would not be a change in the crystal structure in the case of very ductile metals but rather something more complicated. In the case of very ductile metals this would be the ideal situation of the activation of slip systems and the simultaneous advent of the massive flow of dislocations. In the case of more general metals it could be void formation. Even more generally than ductile metals, it could be the formation of weakened regions that still retains solids character. This would be the first indication of damage initiation. Thereafter the damage state would begin to grow and intensify.

This use of the interpretation of yield stress as a 3rd order transition should not be confused with the use of 1st order reaction rate theory in the previous section to solve for the behavior in the strain hardening region. The two separate order designations refer to completely different physical effects.

In the general treatment of failure theory for isotropic materials, Christensen [1], the yield stress was defined as that value of stress at which the magnitude of the second derivative $d^2\sigma / d\varepsilon^2$ attains a maximum. From Fig. 4 it is seen that this previous definition of yield stress is completely compatible with the present definition as a 3rd order transition. In physical reality, it would not be likely that the perfectly idealized 3rd order transition would be or could be realized. Defects and local inhomogeneities would be very likely to "smear" it out into a more smooth, but still rapidly varying form compatible with the yield criterion in Ref. [1]. Nevertheless it is very helpful to see and reveal that the driving force for the appearance of the yield stress is that of moving toward a 3rd order transition.

The end result of these deliberations is that the effective yield stress is that point at which the linear stress-strain behavior deteriorates from linearity into the strain hardening behavior, either because of a 3rd order transition or the driving force moving the material behavior toward that idealized behavior. At the yield stress the constitutive relation changes from that of linear elasticity to that of strain hardening (9).

Implicit in this definition of yield stress for all isotropic materials is that of the existence of a finite failure stress σ_f. This requirement will be considered further in the next two sections.

Ductility and Failure Strain in Idealized Elastic-Perfectly Plastic Solids

In the previous sections the term ductility has been used and appealed to without benefit of definition. Now it is time to be more specific about the meaning of ductility. Considerations will begin with the highly idealized behavior of elastic-perfectly plastic solids, just as it did with the yield stress developments.

The elastic-perfectly plastic case is as shown in Fig. 1. This case of idealized uniaxial tensile behavior must include the possibility of failure. If the strain to failure were quite large then the behavior is as shown in Fig. 5

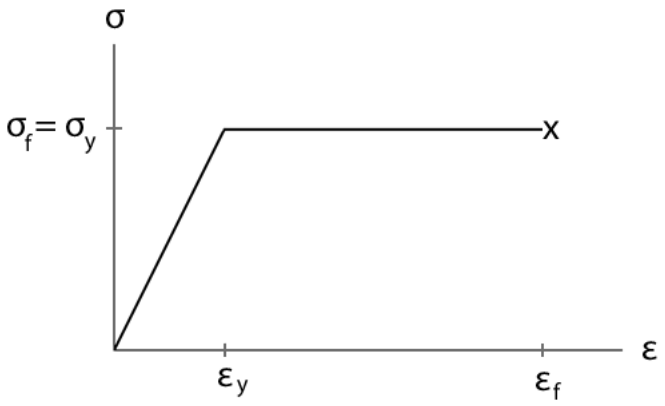

Fig. 5 Extremely ductile failure

Alternatively if the strain to failure was very close to the yield strain then that is brittle failure as shown in Fig. 6.

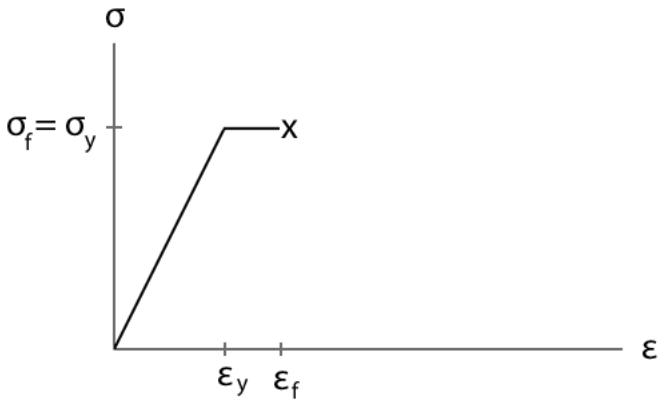

Fig. 6 Extremely brittle failure

Of course the case in Fig. 6 includes the perfectly brittle case with $\varepsilon_f = \varepsilon_y$.

A metric is needed that can distinguish the two cases of Figs. 5 and 6. The only available variable to be used for this purpose is uniaxial strain since only the uniaxial tensile case is being considered. To that end, take the following strain based criterion as a measure of ductility

$$\Lambda = \frac{\varepsilon_f - \varepsilon_y}{\varepsilon_y} \tag{12}$$

It follows that the two cases of Figs. 5 and 6 being considered are specified by

$$\begin{gathered} \Lambda \gg 1 \quad \textit{Extremely ductile} \\ \\ \Lambda \ll 1 \quad \textit{Extremely brittle} \end{gathered} \tag{13}$$

Thus Λ from (12) characterizes the two physical limits and also gives a quantitative measure of ductility for all cases in between. This probably is as much as could be done for ductility when limited to only the one variable, the uniaxial strain in tension. Nevertheless this is valuable information. In fact it is virtually the universal manner used for characterizing ductility and formula (12) is the logical formalization of this method.

Next these results are used to gain entry into the subject of failure. Starting with ductility given by (12) and the two near limiting cases of (13) there is one other case of a ductility measure that is accessible using only (12). This case is that of

$$\Lambda = 1 \quad \textit{Significant ductility} \tag{14}$$

This case is labeled as that of significant ductility because combining (12) and (14) gives

$$\varepsilon_f = 2\varepsilon_y \tag{15}$$

And this case is as shown in Fig. 7.

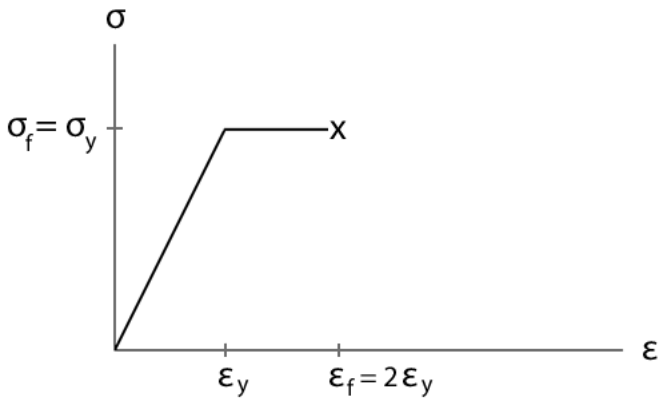

Fig. 7 Significant ductility at failure

If strain were to be decomposed into elastic and plastic parts for the case of (14) and (15) and Fig.7, then the plastic strain is the same size as the elastic strain. This is certainly the situation where the ductility becomes significant and not just negligible.

It could be tempting to try to interpret the case of $\varepsilon_f = 2\varepsilon_y$ as that of the ductile/brittle transition but that would be incorrect. There is no physical basis for interpreting $\varepsilon_f = 2\varepsilon_y$ as a transition in the way that ε_y was shown to be a 3rd order transition, or a close approximation to it. A complete treatment of the ductile/brittle transition requires far more information than is available from just the uniaxial tensile failure, see Ref. [1]. But $\varepsilon_f = 2\varepsilon_y$ may have some special significance for a rational failure interpretation and specification. This possibility will be examined next.

The physical meaning of the strain at $\varepsilon = 2\varepsilon_y$ will be taken as the failure strain at which the material achieves significant ductility before failing. But this is still limited to only the highly idealized case of elastic-perfectly plastic solids. In the next section failure will be given a much more tangible and useful interpretation.

General Definition of Failure Stress and Strain

In the previous section a relationship between ductility and some type of failure designation was examined for the highly idealized case of elastic-perfectly plastic behavior. Also some hypothetical cases of failure occurring at particular strain values were used. Now the concept of failure will be enlarged to represent a failure of function, not just actual physical failure by breakage. This will first be done for the idealized elastic-perfectly plastic case and then thereafter for all of the more general cases.

Fig. 8 again shows the elastic-perfectly plastic case but now the actual physical failure by fragmentation is left unspecified.

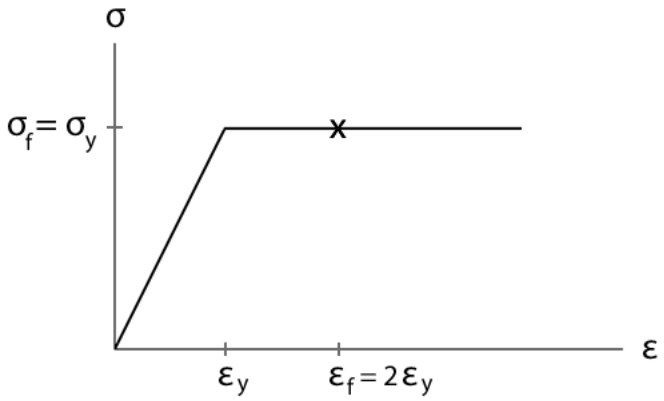

Fig. 8 Failure of function at $\varepsilon_f = 2\varepsilon_y$

The designation of functional failure as occurring at $\varepsilon_f = 2\varepsilon_y$ comes from the preceding section where it was shown that this value of strain is that at which significant or full ductility is first achieved. Beyond that point other extraneous effects could develop. This result follows from the ductility measure given by (12)-(15). It is the concept of effective failure as related to the commonly used measure of ductility according to uniaxial strain.

Now move on to the general case of strain hardening, as in Fig. 3. The obvious and first attempt to assign failure in this general case would be to take the failure strain as $\varepsilon_f = 2\varepsilon_y$ from the elastic-perfectly plastic case and then determine the corresponding failure stress from the experimental

strain hardening curve. Unfortunately this simple procedure would be completely incorrect. The easiest way to justify that realization would be to note that the resulting failure strain would be independent of the degree and type of strain hardening. On further consideration it is realized that both the failure stress and the failure strain must be intimately involved with the full behavior of the strain hardening region and probably also depend on the previous elastic region of behavior as well.

Where does this seeming "roadblock" leave the general failure specification? Return now to the previous elastic-perfectly plastic case in Fig. 9.

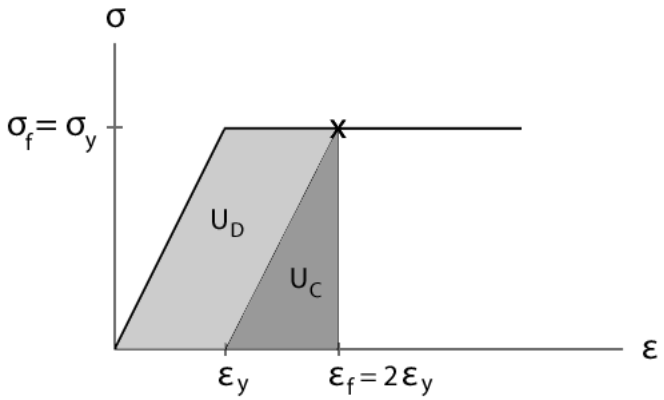

Fig. 9 Conserved and dissipated energies at failure of function

Designate the conserved and dissipated energies as shown in Fig. 9 and as separated by the ideal elastic unloading path. Note that the previously ductility deduced effective failure strain of $\varepsilon_f = 2\varepsilon_y$ is equivalent to requiring that

$$U_D = 2U_C \tag{16}$$

For the elastic-perfectly plastic case, the condition (16) still designates the failure as a failure of function, not as a breakage into separate pieces. The form (16) as a failure criterion will now be rationalized for general application.

With the necessity to screen off failure designations at large strains in order to preclude anomalous effects, take the general case of strain hardening and apply the failure criterion (16) to it as shown in Fig. 10.

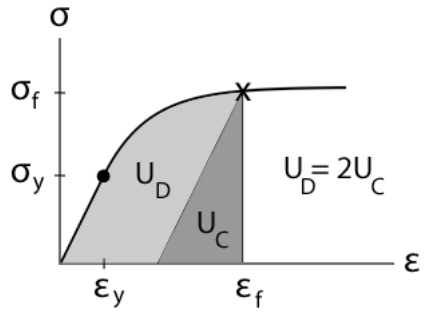

Fig. 10 General failure criterion

In this method of determining effective failure, or simply failure from this point on, the failure stress and failure strain are totally dependent upon the full and complex nature of the strain hardening characteristics.

The failure criterion is formally stated as

At $U_D = 2U_C$

$$\varepsilon = \varepsilon_f$$

$$\sigma = \sigma_f \tag{17}$$

In general, because of the strain hardening effect, the failure strain will be much larger than that of $\varepsilon_f = 2\varepsilon_y$ for the elastic-perfectly plastic case. There could be other criteria that give nearly the same result as (17) in some cases. A simple example is that of $\sigma_f = (E/2)\varepsilon_f$. But the energy form (17) is preferred because of its relationship to ductility.

The general failure criterion given by (17) also reveals the characteristic that for this failure criterion the ductility level indicated by U_D is at the threshold of significant or full ductility.

It is important to observe that if actual physical failure occurs before the strain hardening form satisfies the failure criterion (17) then that level of failure stress and strain is the controlling specification.

The criterion (17) effectively screens off the anomalous behaviors that sometimes show an "upturn" in the strain hardening form that can appear a large strain levels. Such aberrant effects are due to complex anisotropic induced effects and other uniaxial affectations that are not representative of general three dimensional failure.

The failure criterion (17) is identical with that derived in Ref. [1] by a different method. This failure criterion is general, coordinated with but still independent of the yield stress, and certainly explicit and unambiguous.

In the next section a specific and realistic example of strain hardening will be used to demonstrate the efficacy of the failure criterion (17). Also, (17) will be stated in an alternate but equivalent form.

Strain Hardening Test of Failure Stress Definition

To complete the developments, the new strain hardening formulation (5)-(7) will be used to test the failure stress definition that was designed for use with it. A specific but common strain hardening example will be stated and then analyzed for the failure stress and strain identified with it. Before posing the example, the general forms that are needed will be developed.

The failure criterion (17) requires the development of the conserved and dissipated energies U_C and U_D. The conserved energy in Fig. 10 is simply given by

$$U_C = \frac{1}{2}\frac{\sigma^2}{E} \tag{18}$$

Using the strain hardening forms (5)-(7) then gives

$$2EU_C = \sigma_y^2 + 2\sigma_y\sigma_0\left[1 - e^{\alpha(\varepsilon_y - \varepsilon)}\right] + \sigma_0^2\left[1 - e^{\alpha(\varepsilon_y - \varepsilon)}\right]^2 \quad (19)$$

The dissipated energy in Fig. 10 is given by

$$U_D = \int_0^{\varepsilon} \sigma\, d\varepsilon - U_C \quad (20)$$

or

$$U_D = \int_0^{\varepsilon_y} \sigma\, d\varepsilon + \int_{\varepsilon_y}^{\varepsilon} \sigma\, d\varepsilon - U_C \quad (21)$$

This then becomes

$$U_D = \frac{1}{2E}\left(\sigma_y^2 - \sigma^2\right) + \int_{\varepsilon_y}^{\varepsilon} \sigma\, d\varepsilon \quad (22)$$

Carrying out the integration in (22) using (5)-(7) results in

$$U_D = \frac{1}{2E}\left(\sigma_y^2 - \sigma^2\right) + \left(\sigma_y + \sigma_0\right)\left(\varepsilon - \varepsilon_y\right) - \frac{\sigma_0}{\alpha}\left[1 - e^{\alpha(\varepsilon_y - \varepsilon)}\right] \quad (23)$$

Combining σ from (5)-(7) into (23) and then combining U_C and U_D into the failure criterion (17) gives

$$\begin{aligned} &1 + \left(\frac{\sigma_y}{\sigma_0}\right)^2 + \alpha\left(1 + \frac{\sigma_y}{\sigma_0}\right)\left(\varepsilon_y - \varepsilon\right) - e^{\alpha(\varepsilon_y - \varepsilon)} \\ &+ 3\frac{\sigma_y}{\sigma_0}\left[1 - e^{\alpha(\varepsilon_y - \varepsilon)}\right] + \frac{3}{2}\left[1 - e^{\alpha(\varepsilon_y - \varepsilon)}\right]^2 = 0 \end{aligned} \quad (24)$$

When a specific problem is stated then (24) can be solved for the strain at failure $\varepsilon = \varepsilon_f$. Three data parameters are needed to specify (24), ε_y, σ_y / σ_0, and α.

Now the example will be taken as that of a light alloy such as an aluminum form. Take

$$E = 80\ GPa$$

and

$$\sigma_0 = \sigma_y = \frac{\sigma_f}{2}$$

with

$$\sigma_y = 160\ MPa$$
$$\sigma_f = 320\ MPa$$

These then give

$$\varepsilon_y = 0.2x10^{-2}$$
$$\alpha = 500$$

Note that the symbol σ_f will be used for both the failure stress as the asymptote of the strain hardening example and as the result from the failure criterion (17). Whichever one it is will be clear from the context. The above stated value is for the asymptotic value of σ_f.

For this particular example, the failure criterion (24) when stated as percent strain, not absolute strain, becomes

$$17 - 20\varepsilon_f - 14e^{1-5\varepsilon_f} + 3e^{2\left(1-5\varepsilon_f\right)} = 0 \qquad (25)$$

There is one positive root of (25) and it is given by

$$\varepsilon_f = 0.819\%$$

The corresponding failure stress from the strain hardening form (5)-(7) is given by

$$\sigma_f = 313\ MPa$$

This value of the failure stress is at 98% of the asymptotic value of 320MPa. Thus the failure criterion completely captures the failure stress behavior of the light alloy example. The strain at failure is seen to be about twice as large as that from the corresponding elastic-perfectly plastic form of $\varepsilon_f = 2\varepsilon_y$. Fig. 11 shows these results for the light alloy example.

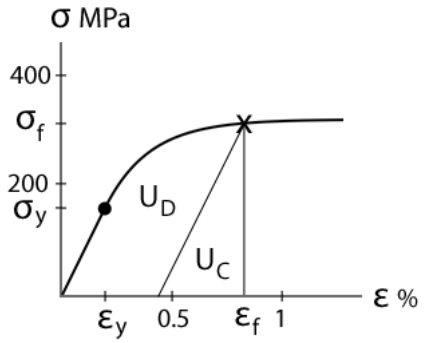

Fig. 11 Light alloy example

The overall conclusion on this failure stress example is that the conserved and dissipated energies based failure criterion (17) shows a well balanced capability for assessing the failure stresses and strains in well posed examples while still preserving the capability to screen off erratic and erroneous data cases that claim to possess very high strength capabilities at large strains.

Noting that the total work done in deforming the material is $W = U_C + U_D$ and using this in the failure criterion (17) gives the alternate form of it as

When

$$\int_0^{\varepsilon} \sigma \, d\varepsilon = \frac{3\sigma^2}{2E} \tag{26}$$

Then

$$\begin{aligned} \sigma &= \sigma_f \\ \varepsilon &= \varepsilon_f \end{aligned} \tag{27}$$

The left had side of (26) is simply the area under the entire stress-strain curve, the work.

Although the presentation in this paper probes the matters of yield stress and failure stress much further than does the treatment in Ref. [1], everything in that reference is completely compatible with the results here.

Finally, although this strain hardening failure example was conducted in full detail, it should also be apparent that failure stresses and strains as specified by (17) are in many or most cases apparent and assignable by inspection of the relevant data sets.

Computational Opportunities and Challenges

There are at least two major reasons why it is so important to develop consistent and reliable methods (and definitions) for determining materials properties. The first is to ensure a safe and reliable data base for assessing the capabilities of different materials and materials classes. The second reason is to actually use these properties in materials behavior applications, sometimes critical applications. The first reason just given is self evident. The second reason involves great complexity and will be discussed further here in the materials failure context.

The major theme of failure theory is to use the minimal failure data mechanical properties to calibrate failure theories that are then capable of predicting the state of safety or that of failure throughout the entire domain

of three dimensional stress space, in all applications. Although this has always been the ultimate goal, it is only now beginning to come within the realm of realization. This is right now occurring through the tremendous leverage that computational capabilities offers for all problems of load bearing structures and through a new and comprehensive theory of failure.

The new theory of failure, Christensen [1], is fully calibrated by only the uniaxial tensile and compressive failure stress values. It applies to all isotropic materials in load bearing structures. Essentially it applies to all engineering materials: brittle metals, ductile metals, glassy polymers, ceramics, glasses, and isotropic minerals. Since the failure actuates in the small strain range for these materials, the corresponding deformation modeling only needs to replicate the small strain range of behaviors. However, these small strain level behaviors can be and often are embedded in kinematical conditions appropriate to large displacements and large rotations and other nonlinearity sources. The corresponding finite element programs in general must therefore employ nonlinear kinematics and effects.

There is a major opportunity to implement the new failure theory into finite element or computational programs under these general conditions. It would be necessary to utilize fully nonlinear forms such as the Cauchy stress or the Piola-Kirchhoff stress forms and the appropriate nonlinear kinematical relations such as the right or left Cauchy-Green tensors. The overall analysis will still locally reduce to small strain conditions before the advent of local failure. Although this plan and procedure may seem straightforward in overall objective, there would be many crucial junctions and options to be examined and tested in order to find the best overall and optimal formulation and approach.

Computational strategies and resources finally has it within its reach to provide a full scale, all inclusive predictive capability extending from initial load application through final materials and structural failure. When this plan is fully developed and widely available it will provide the most powerful tool that could possibly be expected. Until that happens the finite element programs and computational resources will not be fully realized and utilized.

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Richard M. Christensen

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